Vorticity affects the stability of neutron stars

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(February 7, 2008)

The spin rate Ω of neutron stars at a given temperature T is constrained by the interplay between gravitational-radiation instabilities and viscous damping. Navier-Stokes theory has been used to calculate the viscous damping timescales and produce a stability curve for r-modes in the (Ω, T) plane. In Navier-Stokes theory, viscosity is independent of vorticity, but kinetic theory predicts a coupling of vorticity to the shear viscosity. We calculate this coupling and show that it can in principle significantly modify the stability diagram at lower temperatures. As a result, colder stars can remain stable at higher spin rates.

97.60.Jd, 04.40.Dg, 97.10.Sj

It has recently been shown [1,2] that in rotating stars the r-modes are unstable via a coupling to gravitational radiation. This instability can account for the spin-down of hot newly-born neutron stars. Viscosity damps out the oscillations, and tends to stabilize them in general. The timescales associated with this instability have been calculated for uniform density and Newtonian polytropic stellar models up to the lowest order of stellar spin rate Ω [3,4].

r-mode instability could be crucial in understanding the observed spin rates of neutron stars at different temperatures, and it is also a potentially observable source of gravitational radiation [5]. It is therefore important to improve our understanding of the various factors that go into r-mode stability analysis. As examples, the bulk viscous timescale has recently been calculated to second order in Ω [6,7], and relativistic effects on r-modes have recently been considered [8]. Further work remains to be done, both in refining the thermo-hydrodynamical model, and in investigating relativistic r-modes. In this Letter, we consider the possible effects of vorticity on shear viscosity. These effects are predicted by kinetic theory when the unperturbed equilibrium state is rotating.

In standard Navier-Stokes theory, the angular velocity of the fluid has no effect on viscous stress or heat flux, which obey the equations

$$\Pi = -\zeta\Theta\,,\ q_a = -\kappa\nabla_a T\,,\ \pi_{ab} = -2\eta\sigma_{ab}\,, \qquad (1)$$

where Π is the bulk viscous stress and ζ is the bulk viscosity; q_a is the heat flux and κ is the thermal conductivity; π_{ab} is the shear viscous stress and η is the shear viscosity; $\Theta = \nabla_a v^a$ is the volume expansion rate of the fluid, T is the temperature, and $\sigma_{ab} = \nabla_{\langle a}v_{b\rangle}$ is the rate of shear (where the angled brackets denote the symmetric trace-free part). The fluid vorticity $\omega_a = \frac{1}{2}\mathrm{curl}\, v_a$ does not enter these equations, even when the equilibrium state is rotating.

On physical grounds, one might expect that rotational accelerations can couple with gradients of momentum and temperature, so that there could in principle be couplings of ω_a to q_a and π_{ab}. In the case of heat flux, qualitative particle dynamics indicates [9] (p. 34) that this coupling does exist as a result of a Coriolis effect, which is in some sense analogous to the Hall effect in a conductor subject to a magnetic field. The Coriolis effect on heat flux is confirmed by molecular dynamics simulations [10]. Müller [11] and Israel & Stewart [12] showed that the Boltzmann equation predicts in general a coupling of vorticity to heat flux and shear viscous stress. The microscopic and self-consistent kinetic approach is in contrast to the continuum view, where a phenomenological principle of "frame indifference" is invoked to argue against any vorticity coupling. (See [9,10,13] for further discussion.)

Using the Grad moment method to approximate the hydrodynamic regime via kinetic theory, the relations in Eq. (1) are modified to [12] (Eq. (7.1))

$$\Pi = -\zeta\left[\Theta + \beta_0\dot{\Pi}\right]\,, \qquad (2)$$

$$q_a = -\kappa\left[\nabla_a T + T\beta_1\left\{\dot{q}_a - \omega_{ab}q^b\right\}\right]\,, \qquad (3)$$

$$\pi_{ab} = -2\eta\left[\sigma_{ab} + \beta_2\left\{\dot{\pi}_{\langle ab\rangle} - 2\omega^c{}_{\langle a}\pi_{b\rangle c}\right\}\right]\,, \qquad (4)$$

where β_A can be evaluated in terms of collision integrals for specific gases, an overdot denotes the comoving (Lagrangian) derivative, and the vorticity tensor is given by

$$\omega_{ab} = \nabla_{[a}v_{b]} = \varepsilon_{abc}\omega^c\,,$$

where square brackets on indices indicate the skew part. Navier-Stokes theory is recovered from the Müller-Israel-Stewart theory when $\beta_A = 0$. However, kinetic theory gives β_A values for simple gases which are definitely *not* zero. Furthermore, if $\beta_A = 0$, the equilibrium states are unstable and dissipative signals can propagate at unbounded speed [12,13].

The β_A-corrections will be very small except if there are either high frequency oscillations (pumping up the time-derivative terms) or rapid rotation (pumping up the vorticity-coupling terms). In the context of rapidly rotating neutron stars, we expect the vorticity-dissipative couplings to dominate the time-derivative terms; this expectation is borne out by calculations (see below). The vorticity-dissipative couplings will be negligible if the unperturbed equilibrium state is irrotational, i.e., if $\omega_a = 0$ in the background, so that the coupling terms become second-order. However, for fast rotation, $\omega_a \neq 0$ in the background and the coupling terms make a first-order contribution to dissipation. In the words of Israel & Stewart [12]:
"these results will ultimately be of practical interest in astrophysical and cosmological situations involving fast rotation, strong gravitational fields or rapid fluctuations (neutron stars, black hole accretion, early universe), although it will probably be some time before the state of the art in these fields makes such refinements necessary." We believe that recent and ongoing developments in rotating neutron star physics have reached the stage where the Müller-Israel-Stewart theoretical corrections to the Navier-Stokes equations need to be examined, and our results indicate that the corrections could be important.

We follow the standard assumption [14] that the heat flux may be neglected relative to viscous stresses in calculating damping timescales. (Elsewhere we will discuss how the vorticity-heat flux coupling could affect this standard assumption.) Then the vorticity correction to Navier-Stokes theory reduces to the coupling term $\omega^c{}_{\langle a}\pi_{b\rangle c}$. This term means that the angular momentum of the star changes the shear viscosity timescale, and we find (for axial r-modes) a correction proportional to $T^{-r}\Omega^2$, where $r = 9$ for a nonrelativistic fluid and $r = 12$ for an ultrarelativistic fluid.

The evolution of dissipation energy contained in small fluctuations is given by

$$\frac{d\tilde{E}}{dt} = -\int \left[\frac{|\delta\Pi|^2}{\zeta} + \frac{\delta\pi^{ab}\delta\pi^*_{ab}}{2\eta}\right] d^3x - \left(\frac{d\tilde{E}}{dt}\right)_{\mathrm{G}} , \qquad (5)$$

where $(d\tilde{E}/dt)_{\mathrm{G}}$ is the energy flux in gravitational radiation, $\delta\Pi = \Pi - \bar{\Pi}$ and $\delta\pi_{ab} = \pi_{ab} - \bar{\pi}_{ab}$, with an overbar denoting background quantities. In this case, $\bar{\Pi} = 0 = \bar{\pi}_{ab}$. The normal modes of the star are damped by dissipation, and the damping rate can be determined by Eq. (5). For a normal mode with time dependence $e^{i\varpi t}$, the energy has time dependence $\exp[-2\mathrm{Im}(\varpi)t]$. Then by Eq. (5), the characteristic damping time $\tau = 1/\mathrm{Im}(\varpi)$ of the fluid perturbation is given by

$$\frac{1}{\tau} = -\frac{1}{2\tilde{E}}\frac{d\tilde{E}}{dt} = \frac{1}{\tau_{\mathrm{B}}} + \frac{1}{\tau_{\mathrm{S}}} + \frac{1}{\tau_{\mathrm{G}}} , \qquad (6)$$

where τ_{B}, τ_{S}, and τ_{G} are the bulk viscous, shear viscous, and gravitational radiation timescales respectively.

To evaluate the vorticity-corrected shear viscous timescale, we use Eq. (4) in Eqs. (5) and (6). To lowest order

$$\delta\pi_{ab} = -2\eta\left[\delta\sigma_{ab} - 2i\varpi\eta\beta_2\delta\sigma_{ab} + 4\eta\beta_2\delta\sigma^c_{\langle a}\,\omega_{b\rangle c}\right] ,$$

where ω_a is the background vorticity (the background shear vanishes). Then

$$\begin{aligned}\delta\pi^{ab}\delta\pi^*_{ab} = 4\eta^2 \big\{ &\delta\sigma^{ab}\delta\sigma^*_{ab} + 4\gamma^2\left[\varpi^2\delta\sigma^{ab}\delta\sigma^*_{ab}\right.\\ &\left.+ 4\left(\delta\sigma^{ab}\delta\sigma^*_{ab}\omega^c\omega_c - \delta\sigma^{ca}\delta\sigma^*_{da}\omega_c\omega^d\right)\right]\big\} ,\end{aligned}$$

where $\gamma = \eta\beta_2$. The first term is the usual term in Navier-Stokes theory, while the following terms are the Müller-Israel-Stewart corrections. The ϖ^2 term arises from $\dot{\pi}_{ab}$ in Eq. (4), and is negligible relative to the ω^2 terms which arise from the $\omega^c{}_{\langle a}\pi_{b\rangle c}$ term in Eq. (4). The energy dissipation rate through shear viscosity will be

$$\begin{aligned}\left(\frac{d\tilde{E}}{dt}\right)_{\mathrm{S}} = -2\int \eta \big\{ &\delta\sigma^{ab}\delta\sigma^*_{ab} - 4\gamma^2\left[\varpi^2\delta\sigma^{ab}\delta\sigma^*_{ab}\right.\\ &\left.+ 4\left(\delta\sigma^{ab}\delta\sigma^*_{ab}\omega^c\omega_c - \delta\sigma^{ca}\delta\sigma^*_{da}\omega_c\omega^d\right)\right]\big\}\, d^3x. \qquad (7)\end{aligned}$$

In order to proceed further, we need expressions for the shear viscosity η and the coupling coefficient β_2. For the various interactions, $\eta(\rho, T)$ is calculated in [15,16], where it is shown that electron-electron scattering is more important for shear viscosity than other interactions. The expression for η is given in [14], in good agreement with [15,16], as

$$\eta = 1.10 \times 10^{16}\left(\frac{\rho}{10^{14}\mathrm{g/cm}^3}\right)^{9/4}\left(\frac{10^9\mathrm{K}}{T}\right)^2 \mathrm{g/cm\,s}. \qquad (8)$$

For a Maxwell-Boltzmann gas, the coefficient β_2 is found in [12], but we require the expression for a degenerate Fermi gas. This has been found by Olson & Hiscock [17] in the case of strong degeneracy:

$$\beta_2 = \frac{15\pi^2\hbar^3}{m^4 g c^5}\frac{(1+\nu)}{(\nu^2+2\nu)^{5/2}} + \mathcal{O}\left[\left(\frac{kT}{mc^2\nu}\right)^2\right] , \qquad (9)$$

where m is the particle mass, g is the spin weight, and $mc^2\nu/kT \gg 1$. The dimensionless thermodynamic potential $\nu = (\rho + p)/nm - mc^2 s/kT - 1$, where s is the specific entropy, is equal to the nonrelativistic chemical potential per particle divided by the particle rest energy. For a strongly degenerate gas, the nonrelativistic chemical potential is proportional to T, so that

$$\nu \approx \alpha\,\frac{kT}{mc^2} ,$$

where $\alpha \gg 1$ is a dimensionless constant measuring the degree of degeneracy. The nonrelativistic regime is obtained for $\nu \ll 1$, while the ultrarelativistic case corresponds to $\nu \gg 1$.

For temperatures below 10^{10} K, neutrons in the neutron star are nonrelativistic, while electrons are ultrarelativistic [15]. The nonrelativistic limit of β_2 is

$$(\beta_2)_{\rm NR} \approx 3.16 \times 10^{-5} (\alpha T)^{-5/2} \text{ cm s}^2/\text{g}\,, \tag{10}$$

and its ultrarelativistic limit is

$$(\beta_2)_{\rm UR} \approx 6.45 \times 10^{15} (\alpha T)^{-4} \text{ cm s}^2/\text{g}\,. \tag{11}$$

Using Eqs. (8), (10) and (11), we have

$$\gamma_{\rm NR} \approx \frac{1.10 \times 10^{-11}}{\alpha^{5/2}} \left(\frac{\rho}{10^{14}\text{g/cm}^3}\right)^{9/4} \left(\frac{10^9\text{K}}{T}\right)^{9/2} \text{s}\,, \tag{12}$$

$$\gamma_{\rm UR} \approx \frac{7.08 \times 10^{-5}}{\alpha^4} \left(\frac{\rho}{10^{14}\text{g/cm}^3}\right)^{9/4} \left(\frac{10^9\text{K}}{T}\right)^{6} \text{s}\,. \tag{13}$$

In the calculation, we used the same relation for η in both cases, because in the high-density regime ($\rho > 10^{14}$g/cm^3) for both electron-electron scattering and electron-neutron scattering, η is proportional to T^{-2}, with nearly equal proportionality factor [15]. For typical values of the temperature, $T = 10^9$ K, and density, $\rho = 3 \times 10^{14}$ g/cm^3, we find that $\gamma_{\rm UR} \sim \alpha^{-4} \times 10^{-4}$ s, while $\gamma_{\rm NR} \sim \alpha^{-5/2} \times 10^{-10}$ s.

We assume that the background is a uniformly rotating star, so that the equilibrium fluid velocity is $v^a = \Omega \varphi^a$, where φ^a is the rotational Killing vector field [8]. The vorticity vector of the equilibrium state is

$$\vec{\omega} = \frac{\Omega}{2r} \left[\cot\vartheta\,, -1\,, 0\right]\,. \tag{14}$$

The r-modes of rotating barotropic Newtonian stars have Eulerian velocity perturbations given by [4]

$$\delta\vec{v} = CR\Omega \left(\frac{r}{R}\right)^{\ell} \vec{Y}^B_{\ell\ell} \exp(i\varpi t)\,, \tag{15}$$

where C is an arbitrary constant, R is the unperturbed stellar radius, and $\varpi = 2m\Omega/\ell(\ell{+}1)$. The magnetic-type vector spherical harmonics $\vec{Y}^B_{\ell m}$ are defined by

$$\vec{Y}^B_{lm} = \frac{r}{\sqrt{\ell(\ell+1)}} \vec{\nabla} \times \left[r\vec{\nabla} Y_{\ell m}(\vartheta, \varphi)\right]\,. \tag{16}$$

The shear of the perturbed star is given by

$$\delta\sigma_{ab} = \nabla_{\langle a}\delta v_{b\rangle}\,. \tag{17}$$

Substituting Eqs. (14)–(17) into Eq. (7), we find the shear viscosity timescale for $\ell = m$:

$$\frac{1}{\tau_{\rm s}} \approx Q_\ell \left[(\ell-1)(2\ell+1)\int_0^R \eta r^{2\ell}\, dr + \Omega^2 \mathcal{S}_\ell\right]\,, \tag{18}$$

where $Q_\ell^{-1} = \int_0^R \rho r^{2\ell+2}\, dr$. The first term in brackets is in agreement with the expression calculated in [4], and $\mathcal{S}_\ell$ is the correction term:

$$\begin{aligned}\mathcal{S}_\ell \approx\, & 16\frac{(\ell-1)(2\ell+1)}{(\ell+1)^2} U_0 \\ & + \frac{\ell(\ell-2)![(2\ell-1)!!]^2}{(\ell+1)(2\ell-1)(2\ell)!}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\ell-\frac{1}{2})} \times \\ & \times \left[(2\ell^3 - 8\ell^2 - 3\ell - 6)U_2 + 12(\ell^3 - \ell^2 - \ell + 1)U_3 \right. \\ & \left. + 2(4\ell^4 - \ell^3 - 9\ell^2 + 5\ell + 1)U_4\right]\,,\end{aligned} \tag{19}$$

where $U_k(T) \equiv R^k \int_0^R \gamma^2\, \eta r^{2\ell-k}\, dr$.

For the $\ell = 2$ modes, Eqs. (18) and (19) give

$$\begin{aligned}\frac{1}{\tau_{\rm s}} =\, & 5Q_2 \int_0^R \eta r^4\, dr \\ & + \tfrac{1}{9} Q_2 \Omega^2 \left[80U_0 + 93U_2 + 54U_3 - 42U_4\right]\,.\end{aligned} \tag{20}$$

For comparison with previous calculations based on Navier-Stokes viscosity (see, e.g., [7]), we use an $n = 1$ polytrope with mass $M = 1.4M_\odot$ and radius $R = 12.57$ km to evaluate the integrals in Eq. (20). The bulk viscous and gravitational radiation timescales are unaffected by the vorticity correction, and we obtain

$$\begin{aligned}\frac{1}{\tau(\Omega, T)} =\, & \frac{1}{\tilde{\tau}_{\rm G}}\left(\frac{\Omega}{\Omega_{\rm K}}\right)^6 + \frac{1}{\tilde{\tau}_{\rm B}}\left(\frac{T}{10^9\text{K}}\right)^6 \left(\frac{\Omega}{\Omega_{\rm K}}\right)^2 \\ & + \frac{1}{\tilde{\tau}_{\rm S}}\left(\frac{10^9\text{K}}{T}\right)^2 \left[1 + q\alpha^{4-r}\left(\frac{10^9\text{K}}{T}\right)^r \left(\frac{\Omega}{\Omega_{\rm K}}\right)^2\right]\,,\end{aligned} \tag{21}$$

where $\Omega_{\rm K} = \sqrt{\pi G\bar{\rho}}$, which is $\frac{3}{2}$ times the Keplerian (mass-shedding) frequency, and the vorticity correction factors are

$$q = \begin{cases} 1.36 \times 10^{-23}\,, \\ 5.67 \times 10^{-10}\,, \end{cases} \quad r = \begin{cases} 9 & \text{nonrel}\,, \\ 12 & \text{ultrarel}\,. \end{cases} \tag{22}$$

The standard result (see, e.g., [7]) is regained for $q = 0$, with

$$\tilde{\tau}_{\rm G} = -3.26\,\text{s}\,,\ \tilde{\tau}_{\rm B} = 2.01 \times 10^{11}\,\text{s}\,,\ \tilde{\tau}_{\rm S} = 2.52 \times 10^8\,\text{s}\,.$$

We note that the contribution from the $\dot{\pi}_{ab}$ term in Eq. (4) to the q-correction is less than 1% of the contribution from the $\omega^c{}_{\langle a}\pi_{b\rangle c}$ term.

Now we are able to determine from Eq. (21) the critical angular velocity $\Omega_{\rm C}$, defined by $1/\tau(\Omega_{\rm C}, T) = 0$, which governs stability of the star: if $\Omega > \Omega_{\rm C}$, then dissipative damping cannot overcome the gravitational radiation-driven instability. In Fig. 1 we plot $\Omega_{\rm C}/\Omega_{\rm K}$ against temperature T, showing how the vorticity-viscosity coupling affects the standard result (see, e.g., [7]). Electrons are assumed to dominate the shear viscosity, and they are ultrarelativistic over the range of temperatures.

It is clear from Fig. 1 that the vorticity correction is only appreciable at temperatures $T \lesssim 10^8$ K, but that for these lower temperatures, the correction can be large, especially for smaller α. As the degree of degeneracy increases (i.e., with increasing α), the correction is confined to lower and lower temperatures. The effect of the

vorticity-viscosity coupling is to increase the stable region, so that cooler stars can spin at higher rates and remain stable. This may modify recent results [18] which suggest that r-mode instability could stall the spin-up of accreting neutron stars with $T \gtrsim 2 \times 10^5$ K; if the vorticity correction operates, then the stability region is increased, so that spin-up could be more effective, especially for lower degeneracy parameter α.

Of course, our analysis is limited by the fact that we have followed the standard assumption in viscous stability analysis and ignored superfluid effects that will become important at lower temperatures (see, e.g., [19]). Superfluid "friction" effects are thought to prevent f-mode instability, and these effects are likely to be relevant also for r-modes. These effects may strongly alter the vorticity correction effect, a subject which is currently under investigation. In addition, we have used for our r-mode calculations solutions that assume slow rotation. Thus the $\Omega/\Omega_{\rm K} \gtrsim 0.3$ part of Fig. 1 is an extrapolation to high spin rates, in common with previous stability diagrams. Recent calculations of r-modes for rapid rotation [20] should be used in future calculations of the vorticity correction. Since f-modes are unstable at high spin rate, the effect of the vorticity correction on these modes would also be interesting to calculate.

In conclusion, we have shown that the coupling between vorticity and shear viscous stress predicted by kinetic theory can in principle have a significant effect on r-mode instability in neutron stars. The Müller-Israel-Stewart correction of Navier-Stokes theory predicts that colder stars can remain stable at higher spin rates, so that accreting spin-up could be protected from r-mode instability.

Finally, we remark that the vorticity correction to heat flux, as well as the couplings between heat flux and viscous stress predicted by kinetic theory [11,12] [but not shown in Eqs. (2)–(4)], could lead to interesting modifications of the standard stability curve. In particular, the coupling of bulk viscous stress to heat flux could have an effect at *high* temperatures [21].

We are indebted to Nils Andersson and Sharon Morsink for crucial comments, and we thank also Marco Bruni, Luis Herrera, Mehdi Jahan-Miri, and Josep Triginer for helpful discussions. VR was supported by a Royal Society grant.

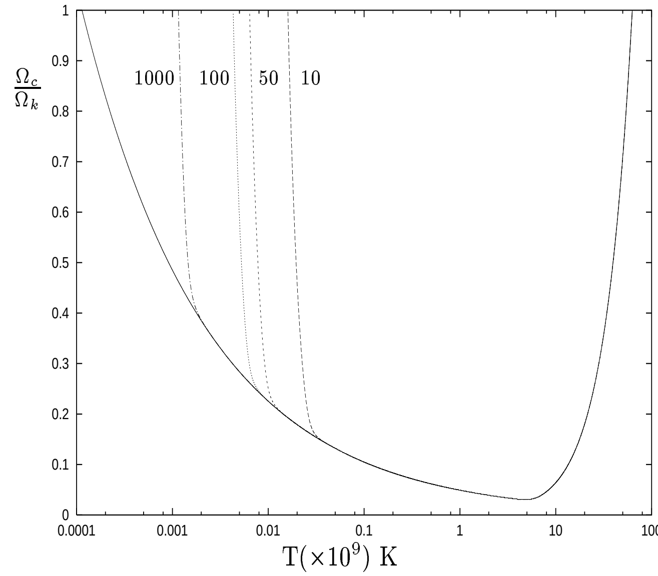

FIG. 1. Critical angular velocity versus temperature ($n = 1$ polytrope with mass $1.4M_\odot$ and radius 12.57 km). The stability region is below the curves. The solid curve shows the standard result, with no coupling of viscosity to vorticity. Broken curves (labelled by the degeneracy parameter α) show how the instability region is reduced by the kinetic-theory coupling of shear viscosity to vorticity, for an ultra-relativistic degenerate Fermi fluid (electron-electron viscosity).

[1] N. Andersson, Astrophys. J. **502**, 708 (1998).
[2] J.L. Friedman and S.M. Morsink, Astrophys. J. **502**, 714 (1998).
[3] K.D. Kokkotas and N. Stergioulas, Astron. Astrophys. **341**, 110 (1998).
[4] L. Lindblom, B.J. Owen, and S.M. Morsink, Phys. Rev. Lett. **80**, 4843 (1998).
[5] B.J. Owen, L. Lindblom, C. Cutler, B.F. Schutz, A. Vecchio, and N. Andersson, Phys. Rev. D **58**, 084020 (1998).
[6] N. Andersson, K.D. Kokkotas, and B.F. Schutz, Astrophys. J. **510**, 846 (1999).
[7] L. Lindblom, G. Mendell, and B.J. Owen, Phys. Rev. D **60**, 064006 (1999).
[8] Y. Kojima, Mon. Not. R. Ast. Soc. **293**, 49 (1998); H.R. Beyer and K.D. Kokkotas, Mon. Not. R. Ast. Soc. **308**, 745 (1999); K.H. Lockitch, Ph.D. thesis, University of Wisconsin-Milwaukee (gr-qc/9909029).
[9] I. Müller and T. Ruggeri, *Extended Thermodynamics* (Springer, Berlin, 1993).
[10] W.G. Hoover, B. Moran, R.M. More, and A.J.C. Ladd, Phys. Rev. A **24**, 2109 (1981).
[11] I. Müller, Arch. Rat. Mech. Anal. **45**, 241 (1972).
[12] W. Israel and J.M. Stewart, Proc. R. Soc. A **365**, 43 (1979).

[13] D. Jou, J. Casas-Vazquez, and G. Lebon, *Extended irreversible thermodynamics* (Springer, Berlin, 1993).
[14] C. Cutler and L. Lindblom, Astrophys. J. **314**, 234 (1987).
[15] E. Flowers and N. Itoh, Astrophys. J. **206**, 218 (1976).
[16] E. Flowers and N. Itoh, Astrophys. J. **230**, 847 (1979).
[17] T.S. Olson and W.A. Hiscock, Phys. Rev. C. **39**, 1818 (1989).
[18] N. Andersson, K.D. Kokkotas, and N. Stergioulas, Astrophys. J., to appear (1999) (astro-ph/9806089); Y. Levin, astro-ph/9810471.
[19] L. Lindblom and G. Mendell, Astrophys. J. **444**, 804 (1995).
[20] L. Lindblom and J. Ipser, Phys. Rev. D **59**, 044009 (1999).
[21] V. Rezania and R. Maartens, in preparation.